J. Fluid Mech. (1982), *vol.* 121, *pp.* 507–515
Printed in Great Britain

The unsteady boundary layer on a rotating disk in a counter-rotating fluid. Part 2

By K. STEWARTSON, C. J. SIMPSON
Department of Mathematics, University College London

AND R. J. BODONYI
Department of Mathematical Sciences,
Indiana–Purdue University at Indianapolis

(Received 9 November 1981)

An unsteady boundary layer on a rotating disk in a counter-rotating fluid was shown in Bodonyi & Stewartson (1977) to develop a singularity at the axis of rotation after a finite time. The structure proposed to describe the singularity is, however, incomplete, even after an additional term was added by Banks & Zaturska (1981). A description is given here in which the boundary layer is divided into three regions; this description seems to be free of the weaknesses of the earlier studies, and is in good agreement with data from the numerical solution of the governing equations.

1. Introduction

In the first paper with this title, Bodonyi & Stewartson (1977, hereinafter referred to as BS) examined the evolution of a laminar boundary layer on a finite rotating disk after its angular velocity Ω, originally the same as that of the ambient fluid, is reversed in sign at time $t^* = 0$. It was shown that the boundary layer develops a singularity at the axis when $\Omega t^* = t_s$, where $t_s \simeq 2{\cdot}36$, and an attempt was made to describe its structure in mathematical terms. However, there were certain discrepancies between the numerical results and the asymptotic expansion about $\Omega t^* = t_s$ which could not be resolved, so that a doubt persisted as to whether a correct description of the structure of the singularity had been given. More recently, Banks & Zaturska (1981, hereinafter referred to as BZ) improved the expansion without altogether removing these doubts. The purpose of the present paper is to show that both analytic descriptions are significantly incomplete, but that the addition of certain new terms and a reformulation of the structure enables a satisfactory correlation with the numerical work to be made.

From the studies in BS it is clear that the singularity develops first at the common axis of rotation of the fluid and the disk, and that in order to investigate its structure only the velocity components of the fluid near this axis need be considered. We define $(\nu/\Omega)^{\frac{1}{2}} z$ to be distance from the disk, where ν is the kinematic viscosity, r the distance from the axis of rotation, and take the velocity components of the fluid in the radial, azimuthal and axial directions to be

$$(\Omega r F_z, \Omega r G, -2(\nu\Omega)^{\frac{1}{2}} F). \tag{1.1}$$

Here F, G are functions of z and $t = \Omega t^*$ only, satisfying the differential equations

$$F_{zt} = F_{zzz} + 2FF_{zz} - F_z^2 + G^2 - 1, \tag{1.2a}$$

$$G_t = G_{zz} + 2FG_z - 2GF_z, \tag{1.2b}$$

where a suffix denotes a derivative. The initial and boundary conditions satisfied by F and G are

$$F = F_z = 0,\ G = 1 \quad \text{at} \quad t = 0 \quad \text{for all} \quad z \geqslant 0, \tag{1.3a}$$

$$G = -1,\ F = F_z = 0 \quad \text{at} \quad z = 0 \quad \text{for all} \quad t > 0, \tag{1.3b}$$

$$G \to 1,\ F_z \to 0 \quad \text{as} \quad z \to \infty \quad \text{for all} \quad t \geqslant 0. \tag{1.3c}$$

2. The modified asymptotic expansion

It was shown in BS that in the numerical solution of (1.2) both G and F_z become large as $t \to t_s$, apparently having asymptotic structures like $(t_s - t)^{-1}$. On this basis an asymptotic expansion about $t = t_s$ was developed for the solution, which, however, contained a serious misconception. This was that it is double-structured, with an inner region extending from the disk, $z = 0$, as far as $z \simeq 2\pi/\alpha(t_s - t)$, where α is a numerical constant. A study of the boundary layer on a heated horizontal cylinder carried out by two of us (Simpson & Stewartson 1982) showed conclusively that the unsteady boundary layer near the highest generator develops a singularity of a similar kind, in a numerical sense, but the structure is triple-layered. Its form is similar to that of a steady suction boundary layer on a rotating disk described by Ockenden (1972) and having a thick inviscid layer sandwiched between two thin viscous layers. The implication for the present problem is that the inner layer is in two parts, and so the previous argument is incomplete. Consequently we shall carry it out *ab initio* and distinguish three regions near the axis when $t_s - t$ is small, in each of which the solution has different properties.

Region I: $z \sim 1$

At finite distances from the disk at $z = 0$ we assume that the solution is smooth for all $t < t_s$ and can be expanded as power series in

$$\tau = t_s - t. \tag{2.1}$$

Thus

$$G = \sum_{n=0}^{\infty} \tau^n G_n(z), \quad F = \sum_{n=0}^{\infty} \tau^n F_n(z), \tag{2.2}$$

where $G_0(0) = -1$, $G_n(0) = 0$ if $n \geqslant 1$, and $F_n(0) = F_n'(0) = 0$ for all n, a prime denoting differentiation with respect to z. On the other hand we cannot expect that $G_0 \to 1$, $F_n' \to 0$ for all n, $G_n \to 0$ for $n \geqslant 1$ as $z \to \infty$, since there are layers of structure above this one when $\tau \ll 1$. It is noted that, according to (2.2), F and G can be found completely in terms of F_0 and G_0; however, we have little information at present about these functions except that their derivatives must satisfy certain compatibility conditions at $z = 0$. The match with the solution in region II will provide additional conditions on F_0 and G_0 as $z \to \infty$, but otherwise F_0 and G_0 must be regarded as arbitrary for the purposes of the analysis. The form of (2.2) implies that the skin-friction components $G_z(\tau, 0)$ and $F_{zz}(\tau, 0)$ are smooth functions of τ, and this is confirmed by the numerical solutions of BS and BZ. Our present calculations also support this conclusion; indeed, although the values are slightly different from those previously given, we do not see the necessity of reporting them in detail.

Region II: $z\tau \sim 1$

For this region, in which the viscous terms are of relatively small importance, we make a slight departure from the notation of BS and write

$$\left.\begin{aligned} \eta &= \alpha\tau z - \alpha\tau g(\tau), \quad G = \tau^{-1}K(\eta,\tau), \\ F &= \frac{1}{\alpha\tau^2}[H(\eta,\tau) - \tfrac{1}{2}\eta + \tfrac{1}{2}\alpha\tau^2 g'(\tau)], \end{aligned}\right\} \tag{2.3}$$

where α is a constant and $g(\tau)$ a function of τ, both of which have to be found. It will emerge later that $g(\tau)/\log\tau$ remains finite as $\tau \to 0$. The equations satisfied by K, H are then

$$-\tau H_{\eta\tau} = 2HH_{\eta\eta} + K^2 - H_\eta^2 + \tfrac{1}{4} - \tau^2 + \alpha^2\tau^3 H_{\eta\eta\eta}, \tag{2.4a}$$

$$-\tau K_\tau = 2HK_\eta - 2H_\eta K + \alpha^2\tau^3 K_{\eta\eta}. \tag{2.4b}$$

The boundary conditions to be satisfied by H, K are that the solution in region II must match smoothly with those in regions I and III. We now assume that H, K may be expanded as series in ascending powers of τ in the form

$$H(\eta,\tau) = H_0(\eta) + \tau H_1(\eta) + \ldots, \tag{2.5a}$$

$$K(\eta,\tau) = K_0(\eta) + \tau K_1(\eta) + \ldots, \tag{2.5b}$$

and allow also for the possibility that at some later stage the expansions may contain powers of $\log\tau$.

In order that the forms (2.2) and (2.5) should blend smoothly across the boundary of regions I and II ($z \to \infty$, $\eta \to 0$), we must have $H_0(0) = 0$, $K_0(0) = 0$, $H_0'(0) = \tfrac{1}{2}$, and so

$$H_0(\eta) = -K_0(\eta) = \tfrac{1}{2}[\sin\eta + \beta(1-\cos\eta)], \tag{2.6}$$

where β is another constant to be found. This result was first derived in BS. Continuing the expansion we find that

$$\begin{aligned} H_1(\eta) = {} & A(1-2H_0') + B(4H_0 + \eta(1-2H_0')) \\ & + C((1-2H_0')\log(1-\cos\eta) + \beta\sin\eta + 2\cos\eta - \beta^2(1-\cos\eta)) \end{aligned} \tag{2.7a}$$

$$K_1(\eta) = 2AH_0' + 2B(\eta H_0' - H_0) + C(2H_0'\log(1-\cos\eta) + \beta^2(1-\cos\eta) - \cos\eta), \tag{2.7b}$$

where A, B, C are constants, and primes denote differentiation with respect to η. BS wrongly assumed that (2.5) holds in both regions I and II, and also set $C = 0$ to exclude the logarithmic singularity at $\eta = 0$. The introduction of region I makes this an unnecessary restriction. The match between regions I and II is effected by expanding H, K in ascending powers of η and then using (2.3) to expand F, G in descending powers of z with $z \gg g(\tau)$. We obtain, for F,

$$\begin{aligned} F \sim {} & +\tfrac{1}{4}\alpha\beta z^2 - 2C\beta z\log z + \left[\frac{2C}{\alpha\tau} + \tfrac{1}{2}g'(\tau)\right] \\ & + z[-\tfrac{1}{2}\alpha\beta g(\tau) - C\beta\log\tfrac{1}{2}\alpha^2\tau^2 - A\beta + 2B + C\beta] + \ldots, \end{aligned} \tag{2.8}$$

the terms omitted being either small when z is large or $O(g^2)$. Thus in order to match with (2.2) we must have

$$g(\tau) + \frac{4C}{\alpha}\log\tau = O(1), \tag{2.9}$$

as $\tau \to 0$, and then

$$F_0(z) = \tfrac{1}{4}\alpha\beta z^2 - 2C\beta z \log z + O(z), \tag{2.10}$$

as $z \to \infty$.

A similar form to (2.8) can be written down for G when τ is small, η is small and $z \gg g(\tau)$. With the requirement that it must match with (2.2), we find that again (2.9) must hold and

$$G_0(z) = -\tfrac{1}{2}\alpha z + 2C\log z + O(1), \tag{2.11}$$

as $z \to \infty$. Both in (2.10) and (2.11) the leading terms omitted can be expressed in terms of A, C, α, β but further terms depend on the higher-order terms of the expansions of H, K in ascending powers of τ and, possibly, powers of $\log \tau$.

Region III: $\eta > 2\pi$

The solution in region II cannot satisfy the boundary condition (1.3*c*) as $z \to \infty$ for $\tau > 0$, and so there must be a region beyond it where F_z and G are both $O(1)$. A match between these two regions must be effected when K, $H - \frac{1}{2}\eta$ are small, i.e. when η is an integral multiple of 2π. The situation is thus similar to that investigated by Ockendon (1972), in the theory of steady flow over a rotating disk. A study of the numerical solution in our problem shows that in fact the transition occurs in the neighbourhood of $\eta = 2\pi$. We therefore write

$$z = \frac{2\pi}{\alpha\tau} + Z, \quad \eta = 2\pi + \bar{\eta}, \tag{2.12}$$

whereupon the forms of H and K near $\bar{\eta} = 0$ are the same, to $O(\tau)$, as those near $\eta = 0$, provided that A is replaced by $A + 2\pi B$. Hence we can expect that if in region III $Z = O(1)$ and $F + \pi/\alpha\tau^2$ and G are expanded, broadly, in powers of τ, as in (2.2), a match can be effected with the solution in region II by taking the double limit $Z \to -\infty$, $\bar{\eta} \to 0$. The use of the word broadly is to permit this expansion to include powers of $\log \tau$ which could reasonably be excluded from (2.2) but not from the expansion when $Z = O(1)$. Using a similar argument to that which led to (2.6), (2.7), we have

$$F = -\frac{\pi}{\alpha\tau^2} + \tilde{F}_0(Z) + o(1), \quad G = \tilde{G}_0(Z) + o(1), \tag{2.13}$$

as $\tau \to 0$ in region III, where $\tilde{F}_0' \to 0$, $\tilde{G}_0 \to 1$ as $Z \to \infty$, and

$$\left.\begin{aligned} \tilde{F}_0 &= \tfrac{1}{4}\alpha\beta Z^2 - 2C\beta Z \log|Z| + O(Z) \\ \tilde{G}_0 &= -\tfrac{1}{2}\alpha Z + 2C\log|Z| + O(1) \end{aligned}\right\} \tag{2.14}$$

as $Z \to -\infty$.

This completes the description of the principal features of the asymptotic solution of the governing equations. BZ noted that a triple-layered solution emerges when higher-order terms are taken into account, but did not elaborate further. They added terms $O(\tau \log \tau)$ to (2.5) in their modification to the expansion proposed by BS, which may be interpreted as equivalent to introducing $g(\tau)$. The term proportional to C in (2.7) was omitted, however, and so their expansion is still incomplete.

3. Numerical methods and results

Two numerical methods were employed to solve the governing equations (1.2), and in both the variable z was replaced by $Y = z/t^{\frac{1}{2}}$ in order to avoid a local singularity at $t = 0$. A full discussion of the solution properties when $t \ll 1$ is given by BZ.

The first method is in essence the same as that used by BS, except for the use of Y as one of the independent variables. It is based on the Crank–Nicholson central-difference scheme and is discussed in detail in their paper. The step length Δt in t was taken as 0·005 for $t \leqslant 2$ and 0·001 for $t > 2$, and uniform step lengths ΔY in Y were used, being either 0·20 or 0·10. In all cases the outer edge of the computational mesh was placed at $Y = 200$. The final results were obtained by h^2 extrapolation.

In the second method the equations were reduced to five first-order partial differential equations in $f = Ft^{\frac{1}{2}}$. $u = \partial f/\partial Y$, $v = \partial u/\partial Y$, G and $k = \partial G/\partial Y$, which were then integrated using the Keller box scheme (Cebeci & Bradshaw 1977). The advantage of this scheme is that a variable step size ΔY in Y is easy to implement, which is helpful near the singularity when it is necessary to extend the range of Y to about 250. In fact the step lengths chosen were $\Delta Y = 0{\cdot}1$ or 0·05 for the first 80 steps, $\Delta Y = 0{\cdot}4$ for the next 40 steps, $\Delta Y = 0{\cdot}8$ for the next 20, and $\Delta Y = 1{\cdot}6$ for the last 150. The step lengths Δt in t were 0·05 for $t < 2$, 0·005 for $2 \leqslant t < 2{\cdot}26$ and 0·001 for $t \geqslant 2{\cdot}26$. Again h^2 extrapolation was used to deduce the data for comparison with the analysis of § 2.

The method used by BZ is similar to our first method, and it is possible to make use of some of their data for comparison with the asymptotic theory.

All these methods lead to data which are close together but not identical. For example at $t = 2{\cdot}25$, the first method yields $F(\infty, t) = -245{\cdot}38$, the second $-248{\cdot}84$, while BZ obtain $-245{\cdot}5$. Incidently, BS obtained $-220{\cdot}9$, but we now regard their results as less accurate than the others and shall discard them. The principal reasons for the differences between the other three methods are, in our view, the errors inherent in any numerical scheme, which is often merely asymptotic, a lack of sufficient care about the extrapolations to zero step size, especially when $t < 2$, and the effect of using a finite outer boundary for Y. We are not able to give an opinion on which set of data is the most accurate. An extraordinary effort would be needed, and we do not believe that any additional insight into the nature of the singularity will result. Only the details of the reduction of the data from the BS scheme will be presented, but we shall include in our discussions some of the results from the other schemes.

The studies by BS and BZ demonstrated that there is almost certainly a singularity at a finite value t_s of t and that the solution structure is close to the form predicted by (2.6). Moreover, the two components of the skin friction are smooth functions of τ in the neighbourhood of $\tau = 0$ and we shall regard this as convincing evidence in favour of the structure proposed for region I. We shall concentrate our attention therefore on region II and begin by setting out in table 1 the extrapolated values of $F(\infty, t)$, $z_{G+}(t)$, the value of z at which G achieves its maximum value $G_{\max}(t)$, and of $z_{G-}(t)$, $G_{\min}$, z_F, $F_{z\min}$ defined in similar ways.

From the analysis of § 2 we see that as $\tau \rightarrow 0$,

$$F(\infty, t) = -\frac{\pi}{\alpha\tau^2} + F_0, \tag{3.1}$$

t	z_{G+}	z_{G-}	z_F	G_{max}	$-G_{min}$	$-F_{z\,min}$	$-F(\infty, t)$
2·0	10·027	1·238	5·509	1·968	1·123	2·772	22·211
2·1	13·904	2·354	7·915	2·686	1·433	3·907	42·973
2·2	23·072	4·749	13·620	4·368	2·241	6·446	114·81
2·25	34·284	7·676	20·651	6·402	3·248	9·455	245·38
2·26	37·927	8·636	22·946	7·059	3·575	10·421	297·80
2·27	42·409	9·822	25·772	7·865	3·978	11·606	368·98
2·28	48·051	11·325	29·337	8·877	4·484	13·091	469·05
2·29	55·365	13·285	33·966	10·186	5·139	15·010	616·07
2·30	65·212	15·939	40·209	11·944	6·019	17·582	844·74
2·31	79·152	19·724	49·070	14·427	7·264	21·215	1228·9
2·32	100·27	25·533	62·596	18·206	9·158	26·731	1949·5

TABLE 1.

t	F_0	M	$-G^*_{max}$	G^*_{min}	$F^*_{z\,min}$
2·0	0·30	0·5706	−0·001	−0·134	0·101
2·1	0·32	0·5336	0·042	−0·062	0·077
2·2	0·34	0·5131	0·082	−0·005	0·051
2·25	0·34	0·5074	0·048	0·018	0·036
2·26	0·34	0·5065	0·101	0·023	0·033
2·27	0·33	0·5058	0·104	0·026	0·030
2·28	0·33	0·5051	0·107	0·030	0·027
2·29	0·34	0·5045	0·109	0·034	0·022
2·30	0·36	0·5040	0·111	0·039	0·019
2·31	0·52	0·5035	0·112	0·042	0·015
2·32	1·28	0·5031	0·109	0·045	0·011

TABLE 2.

where F_0 is strictly $o(\tau^{-1})$ but in fact turns out to have a finite limit. This prediction is a strong test of the theory, but, if successful, should provide good estimates for α and t_s. After some trials we took

$$t_s \simeq 2{\cdot}35851, \quad \alpha \simeq 1{\cdot}0858, \tag{3.2}$$

and the values of F_0 obtained from table 1 are shown in table 2.

It is clear that with these choices for t_s and α, F_0 is approximately equal to 0·34. The use of the second numerical method leads to similar results, except that $\alpha = 1{\cdot}0712$, $t_s = 2{\cdot}35851$ and $F_0 \simeq 0{\cdot}27$. BZ quotes $\alpha \simeq 1{\cdot}09$, $t_s \simeq 2{\cdot}3585$, but we think these numbers may be refined to $\alpha \simeq 1{\cdot}0852$, $t_s \simeq 2{\cdot}35851$, and then $F_0 \simeq 0{\cdot}34$. In the earliest, but less careful, calculation BS obtained $\alpha \simeq 1{\cdot}1$, $t_s \simeq 2{\cdot}365$, but, using table 1 of their paper, we may refine these numbers to $\alpha \simeq 1{\cdot}066$, $t_s \simeq 2{\cdot}364$, and then $F_0 \simeq 0{\cdot}3$.

In order to deduce the value of β we examine the asymptotic forms of G_{max}, G_{min} and $F_{z\,min}$. These follow from (2.5)–(2.8) and are

$$G_{max} = \frac{1}{2\tau}((1+\beta^2)^{\frac{1}{2}} - \beta)(1 + 2(B - C\beta)\tau + \ldots), \tag{3.3a}$$

occurring at

$$\eta_{G+} = -\arctan\frac{1}{\beta} + \pi + O(\tau), \tag{3.3b}$$

$$G_{\min} = -\frac{1}{2\tau}[(1+\beta^2)^{\frac{1}{2}}+\beta][1+2(B-C\beta)\tau+\ldots], \tag{3.4a}$$

occurring at

$$\eta_{G-} = -\arctan\frac{1}{\beta}+O(\tau), \tag{3.4b}$$

and

$$F_{z\min} = -\frac{1}{2\tau}[1+(1+\beta^2)^{\frac{1}{2}}]+(C\beta-B)[(1+\beta^2)^{\frac{1}{2}}-1]+\ldots, \tag{3.5a}$$

occurring at

$$\eta_F = -\arctan\frac{1}{\beta}+\tfrac{1}{2}\pi+O(\tau). \tag{3.5b}$$

Thus a good estimate for β can be obtained by computing

$$M = \frac{-G_{\min}}{G_{\max}} = \frac{(1+\beta^2)^{\frac{1}{2}}+\beta}{(1+\beta^2)^{\frac{1}{2}}-\beta}+O(\tau^2). \tag{3.6}$$

The results are tabulated in table 2; we infer that

$$M = 0{\cdot}5025+0{\cdot}42\tau^2+\ldots, \tag{3.7}$$

fits the data well and leads to

$$\beta \simeq -0{\cdot}3509. \tag{3.8}$$

BZ gave $\beta \simeq -0{\cdot}32$, while our second numerical method leads to $\beta \simeq -0{\cdot}3451$.

These estimates for α, t_s and β may now be tested by computing

$$G^*_{\max} = G_{\max}-\frac{1}{2\tau}[(1+\beta^2)^{\frac{1}{2}}-\beta] = [(1+\beta^2)^{\frac{1}{2}}-\beta](B-C\beta)+O(\tau), \tag{3.9a}$$

$$G^*_{\min} = G_{\min}+\frac{1}{2\tau}[(1+\beta^2)+\beta] = -[(1+\beta^2)^{\frac{1}{2}}+\beta](B-C\beta)+O(\tau), \tag{3.9b}$$

$$F^*_{z\min} = F_{z\min}+\frac{1}{2\tau}[1+(1+\beta^2)^{\frac{1}{2}}] = -[(1+\beta^2)^{\frac{1}{2}}-1][B-C\beta]+O(\tau). \tag{3.9c}$$

These are also set out in table 2. The linear character of the three functions for small values of τ is well brought out, and indeed if we ignore the final row of the table we infer that

$$G^*_{\max} \to -0{\cdot}12, \quad G^*_{\min} \to 0{\cdot}062, \quad F^*_{z\min} \to 0{\cdot}009 \quad \text{as} \quad \tau \to 0. \tag{3.10}$$

If we choose

$$B-\beta C = -0{\cdot}087 \tag{3.11}$$

and make use of (3.8), (3.9), the asymptotic theory predicts the values $-0{\cdot}122$, $0{\cdot}062$, $0{\cdot}005$ respectively for these limits, in good agreement with (3.10).

Finally, we can test the prediction of (2.6) by computing the values of

$$z^*_{G+}(\tau) = -z_{G+}(\tau)+\frac{1}{\alpha\tau}\left(\pi-\arctan\frac{1}{\beta}\right), \tag{3.12a}$$

$$z^*_{G-}(\tau) = -z_{G-}(\tau)-\frac{1}{\alpha\tau}\arctan\frac{1}{\beta}, \tag{3.12b}$$

$$z^*_F(\tau) = -z_F(\tau)+\frac{1}{\alpha\tau}\left(\tfrac{1}{2}\pi-\arctan\frac{1}{\beta}\right). \tag{3.12c}$$

t	z^*_{G+}	z^*_{G-}	z^*_{Fz}
2·0	1·212	1·931	1·695
2·1	1·682	2·040	2·075
2·2	2·348	2·417	2·673
2·25	2·847	2·792	3·148
2·26	2·973	2·894	3·270
2·27	3·113	3·010	3·405
2·28	3·268	3·142	3·556
2·29	3·445	3·294	3·728
2·30	3·650	3·474	3·928
2·31	3·904	3·690	4·165
2·32	4·350	3·960	4·462

TABLE 3.

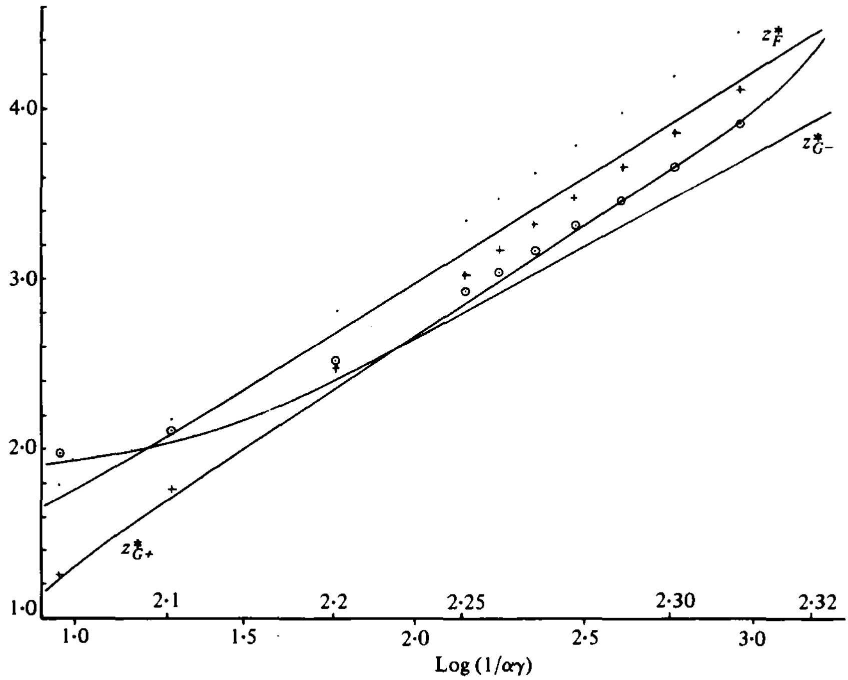

FIGURE 1. Graphs of z^*_{G+}, z^*_{G-} and z^*_F against log $(1/\alpha\tau)$. The solid lines represent the results using the Crank–Nicholson method of integration, and the points marked +, ○, are the corresponding results using the method based on the Keller-box scheme.

According to the asymptotic theory the functions on the left-hand side of (3.12) should asymptote to

$$\frac{4C}{\alpha}\log \alpha\tau + D(\ ) \tag{3.13}$$

as $\tau \to 0$, where the $D(\ )$ are constants with

$$\left.\begin{aligned} D(G+)-D(G-) &= -\frac{2B}{\alpha}\pi - \frac{2C}{\alpha}\log\frac{(1+\beta^2)^{\frac{1}{2}}-\beta}{(1+\beta^2)^{\frac{1}{2}}+\beta}, \\ D(G+)-D(F) &= -\frac{B}{\alpha}\pi - \frac{2C}{\alpha}\log\frac{(1+\beta^2)^{\frac{1}{2}}-\beta}{(1+\beta^2)^{\frac{1}{2}}+1}. \end{aligned}\right\} \tag{3.14}$$

In table 3 we give the values of $z^*(\tau)$, and in figure 1 confirm that they are linear functions of $\log \alpha\tau$ over the range $2{\cdot}25 \leqslant t \leqslant 2{\cdot}31$. The estimated values of $4C/\alpha$ for all three of them are about the same, and approximately equal to $-1{\cdot}3$. Thus $C \simeq -0{\cdot}35$ and $B \simeq 0{\cdot}037$, from (3.11). Further, $D(G+)-D(G-) \simeq 0{\cdot}31$ and $D(G+)-D(F) \simeq -0{\cdot}26$. With $C \simeq -0{\cdot}35$, (3.14) then gives values of 0·024 and 0·005 for B, so that the three estimates are close together. Similar sets of results for $z^*(\tau)$ are obtained by using the second method, and are also displayed in figure 1. Comparisons have been made with the entire profiles of G and F_z (Simpson 1982), and are favourable.

We conclude that the unsteady boundary layer on a rotating disk in a counter-rotating fluid does develop a singularity after a finite time, and that the principal properties of the structure of the singularity are correctly described by the asymptotic expansion in § 2.

REFERENCES

BANKS, W. H. H. & ZATURSKA, M. B. 1981 The unsteady boundary-layer development on a rotating disc in counter-rotating flow. *Acta Mechanica* **38**, 143–145.

BODONYI, R. J. & STEWARTSON, K. 1977 The unsteady boundary layer on a rotating disk in a counter-rotating fluid. *J. Fluid Mech.* **79**, 669–688.

CEBECI, T. & BRADSHAW, P. 1977 *Momentum Transfer in Boundary Layers*. Hemisphere/McGraw-Hill.

OCKENDON, H. 1972 An asymptotic solution for steady flow above an infinite rotating disc with suction. *Q. J. Mech. Appl. Math.* **25**, 291–301.

SIMPSON, C. J. & STEWARTSON, K. 1982 A singularity in an unsteady free-convection boundary layer. *Q. J. Mech. Appl. Math.* (To be published.)

SIMPSON, C. J. 1982 Ph.D. thesis, London University.